

Asymptotic representations and q -oscillator solutions of the graded Yang-Baxter equation related to Baxter Q -operators

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Abstract

We consider a class of asymptotic representations of the Borel subalgebra of the quantum affine superalgebra $U_q(\hat{gl}(M|N))$. This is characterized by Drinfeld rational fractions. In particular, we consider contractions of $U_q(gl(M|N))$ in the FRT formulation and obtain explicit solutions of the graded Yang-Baxter equation in terms of q -oscillator superalgebras. These solutions correspond to L-operators for Baxter Q -operators. We define model independent universal Q -operators as the supertrace of the universal R-matrix and write universal T-operators in terms of these Q -operators based on shift operators on the supercharacters. These include our previous work on $U_q(\hat{sl}(2|1))$ case [1] in part, and also give a cue for operator realization of our Wronskian-like formulas on T- and Q -functions in [2, 3].

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1 Introduction

The Baxter Q -operators were introduced [4] by Baxter when he solved the 8-vertex model. Nowadays his method of Q -operators is recognized as one of the most powerful tools in quantum integrable systems. In particular, Bazhanov, Lukyanov and Zamolodchikov [5] defined Q -operators as the trace of the universal R-matrix over

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q-oscillator representations of the Borel subalgebra of the quantum affine algebra $U_q(\hat{sl}(2))$. Their work based on the q-oscillator algebra was generalized and developed for various directions [6, 7, 8, 9, 1, 10].

In our previous paper [1], we gave Q-operators for the quantum affine superalgebra $U_q(\hat{sl}(2|1))$. Our Q-operators in [1] are universal in the sense that they do not depend on the quantum space and can be applied for both lattice models and quantum field theoretical models as well. We also proposed [2] an idea that there are 2^{M+N} kind of Baxter Q-functions for $U_q(\hat{gl}(M|N))$ case and gave Wronskian-like formulas on T- and Q-functions for finite [2] and infinite [3] dimensional representations for any (M, N) ¹. In this paper, we continue these our previous works and define model independent universal Q-operators for $U_q(\hat{gl}(M|N))$ (or $U_q(\hat{sl}(M|N))$) as the supertrace of the universal R-matrix for any (M, N) . This gives a cue for the operator realization of the Wronskian-like formulas in [2, 3].

In section 2, we define the quantum affine superalgebra (or quantum loop superalgebra) $U_q(\hat{sl}(M|N))$ in terms of the Chevalley generators and the universal R-matrix associated with it. Our task is basically evaluate this universal R-matrix for q-oscillator representations of the Borel subalgebra. As is well known, the Yang-Baxter equation follows from the defining relations of the universal R-matrix. The images of the universal R-matrix for particular representations give the so-called L-operators and R-matrices. The Yang-Baxter equations for the L-operators and the R-matrix ($RLL = LLR$ relations), which are also image of the Yang-Baxter equation for the universal R-matrix, give another realization of the quantum affine superalgebra (FRT realization, [12]). In accordance with the quantum affine superalgebra, the quantum (finite) superalgebra $U_q(gl(M|N))$ also have these two realizations. In section 3, we consider 2^{M+N} kind of contractions of the L-operator for $U_q(gl(M|N))$, which define contracted algebras. A preliminary form of these contractions for $(M, N) = (3, 0)$ case was previously considered in [13]. We also reported such contractions for $(M, N) = (2, 1)$ case in conferences [14].

Next, we consider q-oscillator realizations of these contracted algebras. These induce representations of the Borel subalgebra of the quantum affine superalgebra (or q-superYangian) via the evaluation map. We remark that these representations can not be extended to the full quantum affine superalgebra. These are examples of asymptotic representations characterized by the Drinfeld rational fractions [15]. They are certain limits of the Kirillov-Reshetikhin modules. The heart of an idea is to synchronize the highest weight of the representations and automorphisms of the algebra in the limit so that one can obtain finite quantities. In this way, we obtain spectral parameter dependent L-operators whose matrix elements are written in terms of the q-oscillator superalgebras. Similar L-operators for $(M, N) = (3, 0)$ were previously considered in [16] and [10]. We also reported such L-operators for $(M, N) = (2, 1)$ in [14, 1]. All these L-operators satisfy the defining relations of

¹We also proposed [11] Wronskian-like formulas for infinite dimensional representations for $(M, N) = (4, 4)$ case in the context of the AdS_5/CFT_4 spectral problem.

the universal R-matrix (mentioned in section 2) evaluated by the tensor product of the q-oscillator representations and the fundamental representation of the Borel subalgebras.

In section 4, we define the universal Q-operators as the supertrace of the universal R-matrix over the q-oscillator representations defined in the previous section. The T-operators are written in terms of these Q-operators. In the same way as previous paper [1], our Q-operators here are universal in the sense that they do not depend on the quantum space. As an example, we write Q-operators whose quantum space is the fundamental representation on each lattice site based on the L-operators derived in section 3. Section 5 is devoted to concluding remarks.

There are many literatures on Q-operators related to $sl(2)$, which we could not refer. However there are not so many references for the higher rank case or superalgebras case, which are our main subjects of this paper; and here we only mention some of them for rational models. In the rational limit ($q \rightarrow 1$; after multiplying diagonal matrices for the renormalization), our L-operators naturally reduce to L-operators which are similar to the ones proposed recently in [17] for rational lattice models. However, our L-operators are not simple generalization of the rational ones since many of the non-zero matrix elements of our L-operators become zero in the rational limit. Thus the q-deformation of the rational L-operators is not trivial. There are also Q-operators for infinite dimensional representations on the quantum space [18, 19]. It will be interesting to evaluate our universal Q-operators for infinite dimensional representations on the quantum space and to see how (or if) their formulas are lifted to the trigonometric case. We also proposed [20] Q-operators based on the co-derivative on the supercharacters of $gl(M|N)$. This construction of the Q-operators is useful to discuss [20, 21] functional relations among T- and Q-operators and embed them into the soliton theory. It is desirable to generalize this for the trigonometric case.

2 The quantum affine superalgebra $U_q(\hat{sl}(M|N))$ and the universal R-matrix

Let us introduce a grading parameter $p(i) = 0$ for $i \in \{1, 2, \dots, M\}$ and $p(i) = 1$ for $i \in \{M+1, M+2, \dots, M+N\}$. The quantum affine superalgebra $U_q(\hat{sl}(M|N))$ [22] (see also [23]) is a \mathbb{Z}_2 -graded Hopf algebra generated by the generators² e_i, f_i, h_i, d , where $i \in \{0, 1, \dots, M+N-1\}$. We assign the parity for these generators so that $p(e_0) = p(e_M) = p(f_0) = p(f_M) = 1$ for $MN \neq 0$ and $p(X) = 0$ for all the other generators X . For any $X, Y \in U_q(\hat{sl}(M|N))$, we define $p(XY) = p(X) + p(Y) \pmod{2}$. We introduce the generalized commutator $[X, Y]_q = XY - (-1)^{p(X)p(Y)}qYX$. In particular, we set $[X, Y]_1 = [X, Y]$. For $i, j \in \{0, 1, 2, \dots, M+N-1\}$, the defining

²In this paper, we do not use d .

relations of the algebra $U_q(\hat{sl}(M|N))$ are given by

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad (2.1)$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad (2.2)$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for} \quad a_{ij} = 0, \quad (2.3)$$

where $(a_{ij})_{0 \leq i, j \leq M+N-1}$ is the Cartan matrix $a_{ij} = ((-1)^{p(i)} + (-1)^{p(i+1)})\delta_{ij} - (-1)^{p(i+1)}\delta_{i, j-1} - (-1)^{p(i)}\delta_{i, j+1}$ (here i, j should be interpreted modulo $M+N$: $p(M+N) = p(0)$, $\delta_{i, -1} = \delta_{i, M+N-1}$, $\delta_{i, M+N} = \delta_{i, 0}$). In addition to the above relations, there are Serre relations

$$[e_i, [e_i, e_j]_q]_{q^{-1}} = 0, \quad [f_i, [f_i, f_j]_q]_{q^{-1}} = 0 \quad \text{for} \quad |a_{ij}| = 1, \quad a_{ii} \neq 0, \quad (2.4)$$

$$[e_i, [e_i, [e_i, e_j]_{q^{-2}}]]_{q^2} = 0, \quad [f_i, [f_i, [f_i, f_j]_{q^{-2}}]]_{q^2} = 0$$

$$\text{for} \quad (M, N) = (2, 0), (0, 2), \quad i \neq j, \quad (2.5)$$

and also for the superalgebra case ($MN \neq 0$), the extra Serre relations:

$$[[[e_i, e_j]_{q^{-1}}, e_k]_q, e_j] = 0, \quad [[f_i, f_j]_{q^{-1}}, f_k]_q, f_j] = 0$$

$$\text{for} \quad M+N \geq 5 \quad \text{or} \quad M+N = 4, \quad M \neq N,$$

$$(i, j, k) = (M+N-1, 0, 1), \quad (M-1, M, M+1), \quad (2.6)$$

$$[e_0, [e_2, [e_0, [e_2, e_1]_{q^{-1}}]]]_q = [e_2, [e_0, [e_2, [e_0, e_1]_{q^{-1}}]]]_q,$$

$$[f_0, [f_2, [f_0, [f_2, f_1]_{q^{-1}}]]]_q = [f_2, [f_0, [f_2, [f_0, f_1]_{q^{-1}}]]]_q \quad \text{for} \quad (M, N) = (2, 1), \quad (2.7)$$

$$[e_0, [e_1, [e_0, [e_1, e_2]_{q^{-1}}]]]_q = [e_1, [e_0, [e_1, [e_0, e_2]_{q^{-1}}]]]_q,$$

$$[f_0, [f_1, [f_0, [f_1, f_2]_{q^{-1}}]]]_q = [f_1, [f_0, [f_1, [f_0, f_2]_{q^{-1}}]]]_q \quad \text{for} \quad (M, N) = (1, 2), \quad (2.8)$$

In this paper, we consider the case where the following central element is zero: $h_0 + h_1 + \cdots + h_{M+N-1} = 0$. The algebra has the co-product $\Delta : U_q(\hat{sl}(M|N)) \rightarrow U_q(\hat{sl}(M|N)) \otimes U_q(\hat{sl}(M|N))$ defined by

$$\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad (2.9)$$

$$\Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad (2.10)$$

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad (2.11)$$

where the tensor product is the graded one: $(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)}(AC \otimes BD)$. We assume that every tensor product \otimes in this paper is the graded one. We will also use an opposite co-product defined by

$$\Delta' = \sigma \circ \Delta, \quad \sigma \circ (X \otimes Y) = (-1)^{p(X)p(Y)} Y \otimes X, \quad X, Y \in U_q(\hat{sl}(M|N)). \quad (2.12)$$

In addition to these, there are anti-poisson and co-unit, which will not be used in this paper.

The Borel subalgebras \mathcal{B}_+ (resp. \mathcal{B}_-) is generated by e_i, h_i (resp. f_i, h_i), where $i \in \{0, 1, \dots, M+N-1\}$. Let us take complex numbers $c_i \in \mathbb{C}$ which obey a relation $\sum_{i=0}^{M+N-1} c_i = 0$. Then the following transformation

$$h_i \mapsto h_i + c_i \quad 0 \leq i \leq M+N-1 \quad (2.13)$$

gives a shift automorphism of \mathcal{B}_+ or \mathcal{B}_- . Here we omit the unit element multiplied by the above complex numbers. This automorphism played a role in the construction of the Q-operators in [5, 6, 1]. There exists a unique element [24, 25] $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ called the universal R-matrix which satisfies the following relations

$$\begin{aligned} \Delta'(a) \mathcal{R} &= \mathcal{R} \Delta(a) \quad \text{for } \forall a \in U_q(\hat{sl}(M|N)), \\ (\Delta \otimes 1) \mathcal{R} &= \mathcal{R}^{13} \mathcal{R}^{23}, \\ (1 \otimes \Delta) \mathcal{R} &= \mathcal{R}^{13} \mathcal{R}^{12} \end{aligned} \quad (2.14)$$

where $\mathcal{R}^{12} = \mathcal{R} \otimes 1$, $\mathcal{R}^{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}^{13} = (\sigma \otimes 1) \mathcal{R}^{23}$. The (graded) Yang-Baxter equation

$$\mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{R}^{13} \mathcal{R}^{12}, \quad (2.15)$$

is a corollary of these relations (2.14). The universal R-matrix can be written in the form

$$\mathcal{R} = \overline{\mathcal{R}} q^{\mathcal{K}}, \quad \mathcal{K} = \sum_{i,j=1}^{M+N-1} d_{ij} h_i \otimes h_j, \quad (2.16)$$

where $(d_{ij})_{1 \leq i,j \leq M+N-1}$ is the inverse of the Cartan matrix $(a_{ij})_{1 \leq i,j \leq M+N-1}$ of $sl(M|N)$. In case this Cartan matrix is degenerated ($M = N$), we have to consider an extended matrix and take the inverse of it [26]. Here $\overline{\mathcal{R}}$ is the reduced universal R-matrix, which is a series in $e_j \otimes 1$ and $1 \otimes f_j$ and does not contain Cartan elements. Thus the reduced universal R-matrix is unchanged under the shift automorphism (2.13), while the prefactor of the universal R-matrix (2.16) is shifted as

$$\mathcal{K} \mapsto \mathcal{K} + \sum_{i,j=1}^{M+N-1} d_{ij} c_i (1 \otimes h_j), \quad (2.17)$$

where we considered a shift on \mathcal{B}_+ .

There is a (finite) quantum superalgebra $U_q(gl(M|N))$, which is generated by the elements $\{e_{ij}\}_{i,j=1}^{M+N}$. We assign the parity of these generators as $p(e_{ij}) = p(i) +$

$p(j) \bmod 2$. Let us introduce the notations: $e_{\alpha_i} = e_{i,i+1}$, $e_{-\alpha_i} = e_{i+1,i}$ for $i \in \{1, 2, \dots, M+N-1\}$. Then the defining relations of $U_q(gl(M|N))$ are

$$\begin{aligned}
[e_{ii}, e_{jj}] &= 0, \quad [e_{ii}, e_{\pm\alpha_j}] = \pm(\delta_{i,j} - \delta_{i,j+1})e_{\pm\alpha_j}, \\
[e_{\alpha_i}, e_{-\alpha_j}] &= (-1)^{p(i)}\delta_{ij} \frac{q^{(-1)^{p(i)}e_{ii} - (-1)^{p(i+1)}e_{i+1,i+1}} - q^{(-1)^{p(i)}e_{ii} + (-1)^{p(i+1)}e_{i+1,i+1}}}{q - q^{-1}}, \\
[e_{\alpha_i}, e_{\alpha_j}] &= [e_{-\alpha_i}, e_{-\alpha_j}] = 0 \quad \text{for } |i-j| \geq 2, \\
[e_{\alpha_i}, [e_{\alpha_i}, e_{\alpha_j}]_q]_{q^{-1}} &= [e_{\alpha_i}, [e_{\alpha_i}, e_{\alpha_j}]_q]_{q^{-1}} = 0 \quad \text{for } |i-j| = 1, \\
(e_{\pm\alpha_i})^2 &= 0 \quad \text{for } p(i) = 1.
\end{aligned} \tag{2.18}$$

The other elements are defined by

$$\begin{aligned}
e_{ij} &= [e_{ik}, e_{kj}]_{q^{(-1)^{p(k)}}} \quad \text{for } i > k > j, \\
e_{ij} &= [e_{ik}, e_{kj}]_{q^{-(-1)^{p(k)}}} \quad \text{for } i < k < j.
\end{aligned} \tag{2.19}$$

There are also additional Serre relations. Let E_{ij} be a $(M+N) \times (M+N)$ matrix unit whose (k, l) -element is $\delta_{i,k}\delta_{j,l}$. $\pi(e_{ij}) = E_{ij}$ gives the fundamental representation of $U_q(gl(M|N))$. There is an evaluation map $\mathbf{ev}_x: U_q(\hat{sl}(M|N)) \mapsto U_q(gl(M|N))$:

$$\begin{aligned}
e_0 &\mapsto xq^{(-1)^{p(1)}e_{11}}e_{M+N,1}q^{(-1)^{p(M+N)}e_{M+N,M+N}}, \\
f_0 &\mapsto (-1)^{p(M+N)}x^{-1}q^{(-1)^{p(M+N)}e_{M+N,M+N}}e_{1,M+N}q^{(-1)^{p(1)}e_{1,1}}, \\
h_0 &\mapsto (-1)^{p(M+N)}e_{M+N,M+N} - (-1)^{p(1)}e_{1,1}, \\
e_i &\mapsto e_{i,i+1}, \quad f_i \mapsto (-1)^{p(i)}e_{i+1,i}, \quad h_i \mapsto (-1)^{p(i)}e_{ii} - (-1)^{p(i+1)}e_{i+1,i+1} \\
&\quad \text{for } 1 \leq i \leq M+N-1,
\end{aligned} \tag{2.20}$$

where $x \in \mathbb{C}$ is a spectral parameter. Let π_λ be an irreducible representation of $U_q(gl(M|N))$ with the highest weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{M+N})$ and the highest weight vector $|\lambda\rangle$ defined by

$$e_{ii}|\lambda\rangle = \lambda_i|\lambda\rangle, \quad e_{jk}|\lambda\rangle = 0 \quad \text{for } j < k, \quad i, j, k \in \{1, 2, \dots, M+N\}. \tag{2.21}$$

Then the composition $\pi_\lambda(x) = \pi_\lambda \circ \mathbf{ev}_x$ gives an evaluation representation of $U_q(\hat{sl}(M|N))$. For the fundamental representation, we will use a notation $\pi(x) = \pi_{(1,0,\dots,0)}(x)$. We also use a notation $\pi_\lambda^+(x)$ for the evaluation representation based on the Verma module defined by the free action of the generators on the highest weight vector (2.21). In this case, the representation is not necessary irreducible. Our main task is basically to evaluate the universal R-matrix for various representations of $U_q(\hat{sl}(M|N))$ (or $U_q(\hat{gl}(M|N))$). Namely, to find matrices of the form (2.16) which satisfy (2.14) for various representations of \mathcal{B}_+ and \mathcal{B}_- . The simplest example is the R-matrix for the Perk-Schultz model [27] (see [28] for $N=0$ case), which is a multi-component generalization of the six-vertex model. Namely, the image of

the universal R-matrix for $\pi(x_1) \otimes \pi(x_2)$ gives (up to an overall factor $N(x_1, x_2)$; $x_1, x_2 \in \mathbb{C}$):

$$\mathbf{R}(x_1, x_2) = N(x_1, x_2)(\pi(x_1) \otimes \pi(x_2))\mathcal{R} = \mathbf{R} - \frac{x_1}{x_2}\overline{\mathbf{R}}, \quad (2.22)$$

$$\mathbf{R} = \sum_{i=1}^{M+N} q^{1-2p(i)} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} (-1)^{p(j)} E_{ij} \otimes E_{ji}, \quad (2.23)$$

$$\overline{\mathbf{R}} = \sum_{i=1}^{M+N} q^{-1+2p(i)} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i > j} (-1)^{p(j)} E_{ij} \otimes E_{ji}. \quad (2.24)$$

This obeys the graded Yang-Baxter equation

$$\mathbf{R}^{12}(x_1, x_2)\mathbf{R}^{13}(x_1, x_3)\mathbf{R}^{23}(x_2, x_3) = \mathbf{R}^{23}(x_2, x_3)\mathbf{R}^{13}(x_1, x_3)\mathbf{R}^{12}(x_1, x_2), \quad (2.25)$$

which is an image of (2.15) for $\pi(x_1) \otimes \pi(x_2) \otimes \pi(x_3)$, where $x_1, x_2, x_3 \in \mathbb{C}$.

3 L-operators from FRT realization of the quantum affine superalgebra $U_q(\hat{gl}(M|N))$

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ (and its subalgebra $U_q(gl(M|N))$) has another realization, called FRT realization [12] (see also, [29]), based on the Yang-Baxter equation ($RLL = LLR$ relation). In this section we use this realization. The (centerless) quantum affine superalgebra $U_q(\hat{gl}(M|N))$ is defined by

$$L_{ij}^{(0)} = \overline{L}_{ji}^{(0)} = 0, \quad \text{for } 1 \leq i < j \leq M + N \quad (3.1)$$

$$L_{ii}^{(0)} \overline{L}_{ii}^{(0)} = \overline{L}_{ii}^{(0)} L_{ii}^{(0)} = 1 \quad \text{for } 1 \leq i \leq M + N, \quad (3.2)$$

$$\mathbf{R}^{23}(x, y)\mathbf{L}^{13}(y)\mathbf{L}^{12}(x) = \mathbf{L}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x, y), \quad (3.3)$$

$$\mathbf{R}^{23}(x, y)\overline{\mathbf{L}}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\overline{\mathbf{L}}^{13}(y)\mathbf{R}^{23}(x, y), \quad (3.4)$$

$$\mathbf{R}^{23}(x, y)\mathbf{L}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x, y), \quad (3.5)$$

where $x, y \in \mathbb{C}$ and

$$\mathbf{L}(x) = \sum_{i,j} L_{ij}(x) \otimes E_{ij}, \quad \overline{\mathbf{L}}(x) = \sum_{i,j} \overline{L}_{ij}(x) \otimes E_{ij}, \quad (3.6)$$

and

$$L_{ij}(x) = \sum_{n=0}^{\infty} L_{ij}^{(n)} x^{-n}, \quad \overline{L}_{ij}(x) = \sum_{n=0}^{\infty} \overline{L}_{ij}^{(n)} x^n. \quad (3.7)$$

The above relations came from the graded Yang-Baxter equation (2.15) for the universal R-matrix under the specialization (2.22) and $\mathbf{L}(x) = N(x)(1 \otimes \pi(x))\mathcal{R}$, $\overline{\mathbf{L}}(x) = \overline{N}(x)(1 \otimes \pi(x))(\mathcal{R}^{21})^{-1}$, where $N(x)$ and $\overline{N}(x)$ are overall factors. To be precise, in order to obtain the defining relations for $U_q(\widehat{sl}(M|N))$, we will have to impose a condition that the quantum super-determinants of the above L-operators are 1. But we do not impose these explicitly here. Let us introduce a function: $\theta(\text{True}) = 1$, $\theta(\text{False}) = 0$. One can rewrite (3.3) as

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} (q^{(2p(a)-1)\delta_{ac}}x - q^{(1-2p(a))\delta_{ac}}y) L_{cd}(y) L_{ab}(x) \\ & - (-1)^{(p(a)+p(b))p(d)} (q^{(2p(b)-1)\delta_{bd}}x - q^{(1-2p(b))\delta_{bd}}y) L_{ab}(x) L_{cd}(y) = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) [(\theta(a > c)x + \theta(a < c)y) L_{ad}(y) L_{cb}(x) \\ & \quad - (\theta(d > b)x + \theta(d < b)y) L_{ad}(x) L_{cb}(y)], \end{aligned} \quad (3.8)$$

and (3.4) as

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} (q^{(2p(a)-1)\delta_{ac}}x - q^{(1-2p(a))\delta_{ac}}y) \overline{L}_{cd}(y) \overline{L}_{ab}(x) \\ & - (-1)^{(p(a)+p(b))p(d)} (q^{(2p(b)-1)\delta_{bd}}x - q^{(1-2p(b))\delta_{bd}}y) \overline{L}_{ab}(x) \overline{L}_{cd}(y) = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) [(\theta(a > c)x + \theta(a < c)y) \overline{L}_{ad}(y) \overline{L}_{cb}(x) \\ & \quad - (\theta(d > b)x + \theta(d < b)y) \overline{L}_{ad}(x) \overline{L}_{cb}(y)], \end{aligned} \quad (3.9)$$

and (3.5) as

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} (q^{(2p(a)-1)\delta_{ac}}x - q^{(1-2p(a))\delta_{ac}}y) L_{cd}(y) \overline{L}_{ab}(x) \\ & - (-1)^{(p(a)+p(b))p(d)} (q^{(2p(b)-1)\delta_{bd}}x - q^{(1-2p(b))\delta_{bd}}y) \overline{L}_{ab}(x) L_{cd}(y) = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) [(\theta(a > c)x + \theta(a < c)y) L_{ad}(y) \overline{L}_{cb}(x) \\ & \quad - (\theta(d > b)x + \theta(d < b)y) \overline{L}_{ad}(x) L_{cb}(y)]. \end{aligned} \quad (3.10)$$

For any $c \in \mathbb{C} \setminus \{0\}$,

$$\mathbf{L}(x) \mapsto \mathbf{L}(cx), \quad \overline{\mathbf{L}}(x) \mapsto \overline{\mathbf{L}}(cx) \quad (3.11)$$

gives an automorphism of $U_q(\widehat{gl}(M|N))$ since $\mathbf{R}(cx_1, cx_2) = \mathbf{R}(x_1, x_2)$. The restriction of the relations (3.1)-(3.5) to the relation for $\mathbf{L}(x)$ defines a sort of Borel subalgebra of $U_q(\widehat{gl}(M|N))$ called q -super-Yangian. Note that the following transformation (multiplication of diagonal matrices in the second space)

$$\begin{aligned} \mathbf{L}(x) & \mapsto (1 \otimes \mathcal{H}_L) \mathbf{L}(x) (1 \otimes \mathcal{H}_R), \quad \overline{\mathbf{L}}(x) \mapsto (1 \otimes \mathcal{H}_L) \overline{\mathbf{L}}(x) (1 \otimes \mathcal{H}_R), \\ \mathcal{H}_L & = \sum_i \mathcal{H}_L^{(i)} E_{ii}, \quad \mathcal{H}_R = \sum_i \mathcal{H}_R^{(i)} E_{ii}, \quad \mathcal{H}_L^{(i)}, \mathcal{H}_R^{(i)} \in \mathbb{C} \end{aligned} \quad (3.12)$$

keeps the relations (3.1) and (3.3)-(3.5) (this came from the first relation in (2.14)). However it changes (3.2) as

$$L_{ii}^{(0)} \bar{L}_{ii}^{(0)} = \bar{L}_{ii}^{(0)} L_{ii}^{(0)} = (\mathcal{H}_L^{(i)} \mathcal{H}_R^{(i)})^2 \quad \text{for } 1 \leq i \leq M+N. \quad (3.13)$$

The restriction of this transformation to the q-superYangian gives an automorphism of it. In addition, if we consider a ‘bigger’ algebra (a kind of an asymptotic algebra [15]) which does not assume (3.2), it can be an automorphism of such algebra.

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ has a finite subalgebra $U_q(gl(M|N))$ defined by

$$L_{ij} = \bar{L}_{ji} = 0, \quad \text{for } 1 \leq i < j \leq M+N \quad (3.14)$$

$$L_{ii} \bar{L}_{ii} = \bar{L}_{ii} L_{ii} = 1 \quad \text{for } 1 \leq i \leq M+N, \quad (3.15)$$

$$\mathbf{R}^{23} \mathbf{L}^{13} \mathbf{L}^{12} = \mathbf{L}^{12} \mathbf{L}^{13} \mathbf{R}^{23}, \quad (3.16)$$

$$\mathbf{R}^{23} \bar{\mathbf{L}}^{13} \bar{\mathbf{L}}^{12} = \bar{\mathbf{L}}^{12} \bar{\mathbf{L}}^{13} \mathbf{R}^{23}, \quad (3.17)$$

$$\mathbf{R}^{23} \mathbf{L}^{13} \bar{\mathbf{L}}^{12} = \bar{\mathbf{L}}^{12} \mathbf{L}^{13} \mathbf{R}^{23}, \quad (3.18)$$

where

$$\mathbf{L} = \sum_{i,j} L_{ij} \otimes E_{ij}, \quad \bar{\mathbf{L}} = \sum_{i,j} \bar{L}_{ij} \otimes E_{ij}. \quad (3.19)$$

Then the relation (3.16) leads

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} L_{cd} L_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} L_{ab} L_{cd} = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) - \theta(a < c)) L_{ad} L_{cb}, \end{aligned} \quad (3.20)$$

the relation (3.17) leads

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} \bar{L}_{cd} \bar{L}_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} \bar{L}_{ab} \bar{L}_{cd} = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) - \theta(a < c)) \bar{L}_{ad} \bar{L}_{cb}, \end{aligned} \quad (3.21)$$

and the relation (3.18) leads

$$\begin{aligned} & (-1)^{(p(a)+p(b))p(c)} q^{(1-2p(a))\delta_{ac}} L_{cd} \bar{L}_{ab} - (-1)^{(p(a)+p(b))p(d)} q^{(1-2p(b))\delta_{bd}} \bar{L}_{ab} L_{cd} = \\ & = (-1)^{p(a)p(b)} (q - q^{-1}) (\theta(d < b) \bar{L}_{ad} L_{cb} - \theta(a < c) L_{ad} \bar{L}_{cb}). \end{aligned} \quad (3.22)$$

These generators are related to the generators $\{e_{ij}\}$ in section 2 as

$$L_{ii} = q^{(-1)^{p(i)} e_{ii}}, \quad \bar{L}_{ii} = q^{-(-1)^{p(i)} e_{ii}}, \quad (3.23)$$

$$L_{ij} = (-1)^{p(i)} (q - q^{-1}) e_{ji} q^{(-1)^{p(j)} e_{jj}} \quad \text{for } i > j, \quad (3.24)$$

$$\bar{L}_{ij} = -(-1)^{p(i)} (q - q^{-1}) q^{-(-1)^{p(i)} e_{ii}} e_{ji} \quad \text{for } i < j, \quad (3.25)$$

Then the action of generators on the highest weight vector corresponding to (2.21) is

$$\begin{aligned} L_{ii}|\lambda\rangle &= q^{(-1)^{p(i)}\lambda_i}|\lambda\rangle, \quad \bar{L}_{ii}|\lambda\rangle = q^{-(-1)^{p(i)}\lambda_i}|\lambda\rangle \quad \text{for } 1 \leq i \leq M+N, \\ L_{kj}|\lambda\rangle &= 0 \quad \text{for } 1 \leq j < k \leq M+N. \end{aligned} \quad (3.26)$$

There is an evaluation map from $U_q(\hat{gl}(M|N))$ to $U_q(gl(M|N))$ such that

$$\mathbf{L}(x) \mapsto \mathbf{L} - \bar{\mathbf{L}}x^{-1}, \quad (3.27)$$

$$\bar{\mathbf{L}}(x) \mapsto \bar{\mathbf{L}} - \mathbf{L}x. \quad (3.28)$$

Apparently, the difference between $\mathbf{L}(x)$ and $\bar{\mathbf{L}}(x)$ are not very important under the evaluation map. Let us consider an irreducible representation of $U_q(\hat{gl}(M|N))$ with the highest weight $(\nu(x), \bar{\nu}(x))$ and the highest weight vector $|\nu, \bar{\nu}\rangle$ defined by

$$L_{ii}(x)|\nu, \bar{\nu}\rangle = \nu_i(x)|\nu, \bar{\nu}\rangle, \quad \bar{L}_{ii}(x)|\nu, \bar{\nu}\rangle = \bar{\nu}_i(x)|\nu, \bar{\nu}\rangle \quad \text{for } 1 \leq i \leq M+N, \quad (3.29)$$

$$L_{ij}(x)|\nu, \bar{\nu}\rangle = 0, \quad \bar{L}_{ij}(x)|\nu, \bar{\nu}\rangle = 0 \quad \text{for } i > j, \quad (3.30)$$

where $\nu(x) = (\nu_1(x), \nu_2(x), \dots, \nu_{M+N}(x))$, $\bar{\nu}(x) = (\bar{\nu}_1(x), \bar{\nu}_2(x), \dots, \bar{\nu}_{M+N}(x))$ are tuples of formal power series in x^{-1} and x respectively. For the evaluation representation based on (3.26)-(3.28), (3.29) becomes

$$L_{ii}(x)|\lambda\rangle = (q^{(-1)^{p(i)}\lambda_i} - x^{-1}q^{-(-1)^{p(i)}\lambda_i})|\lambda\rangle, \quad (3.31)$$

$$\bar{L}_{ii}(x)|\lambda\rangle = (q^{-(-1)^{p(i)}\lambda_i} - xq^{(-1)^{p(i)}\lambda_i})|\lambda\rangle \quad \text{for } 1 \leq i \leq M+N. \quad (3.32)$$

For the finite dimensional representations, there exist monic polynomials in x , called Drinfeld polynomials $P_i(x)$, such that

$$\frac{\nu_i(x)}{\nu_{i+1}(x)} = q^{-(-1)^{p(i)}\deg P_i(x)} \frac{P_i(xq^{2(-1)^{p(i)}\lambda_i})}{P_i(x)} = \frac{\bar{\nu}_i(x)}{\bar{\nu}_{i+1}(x)} \quad \text{for } 1 \leq i \leq M+N-1. \quad (3.33)$$

For the evaluation modules whose highest weights are given by (3.31) and (3.32), the Drinfeld polynomials have the form (if $\lambda_i - (-1)^{p(i)+p(i+1)}\lambda_{i+1} \in \mathbb{Z}_{\geq 0}$)

$$P_i(x) = \prod_{k=1}^{\lambda_i - (-1)^{p(i)+p(i+1)}\lambda_{i+1}} (1 - xq^{2(-1)^{p(i+1)}\lambda_{i+1} + 2(-1)^{p(i)}(k-1)}) \quad \text{for } 1 \leq i \leq M+N-1. \quad (3.34)$$

Let us take a subset I of the set $\{1, 2, \dots, M+N\}$ and its complement set $\bar{I} := \{1, 2, \dots, M+N\} \setminus I$. There are 2^{M+N} choices of the subsets in this case. Corresponding to the set I , we consider 2^{M+N} kind of representations of

the q -superYangian. For this purpose, we consider 2^{M+N} kind of contractions of $U_q(gl(M|N))$. Namely, let us consider an algebra whose condition (3.15) is replaced by

$$L_{ii}\bar{L}_{ii} = \bar{L}_{ii}L_{ii} = 1 \quad \text{for } i \in I, \quad (3.35)$$

$$\bar{L}_{ii} = 0 \quad \text{for } i \in \bar{I}. \quad (3.36)$$

Then one can obtain 2^{M+N} kind of algebraic solutions of the graded Yang-Baxter equation via the map (3.27). In addition to the contraction (3.36), we consider the following subsidiary contraction. Suppose the set I has the form $I = \{k+1, k+2, \dots, k+n\}$ for some $k \geq 0, n > 0$, then we assume

$$L_{ij} = 0 \quad \text{for } k+n < i \leq M+N \quad \text{and} \quad 1 \leq j \leq k, \quad (3.37)$$

$$\bar{L}_{ij} = 0 \quad \text{for } 1 < i < j \leq k \quad \text{or} \quad k+n < i < j \leq M+N. \quad (3.38)$$

One may consider different contractions than (3.37),(3.38). Here we consider a contraction so that the location of the zeros becomes cyclic with respect to the shift of the suffixes by an operation: $a \mapsto a+1$ for $a < M+N$ and $M+N \mapsto 1$. Namely, the contraction for $k > 0$ can be given by applying this operation k -times for the case $k = 0$. What is important here is to respect the commutation relations for the generators (3.20)-(3.22). We remark that these contractions on the L-operator for $U_q(\hat{gl}(3))$ (written in terms of the generators e_{ij} and substituted into (3.27)) was previously considered in [13]. We also reported these contractions for $U_q(\hat{gl}(2|1))$ in conferences [14].

The next task is to consider representations of these contracted algebras. We are interested in q -oscillator representations. The q -oscillator (super)algebra (see for example, [30]) is generated by the generators $\mathbf{c}_{ai}, \mathbf{c}_{ai}^\dagger, \mathbf{n}_{ia}$ for $i \in I, a \in \bar{I}$, whose parities are defined by $p(\mathbf{c}_{ai}) = p(\mathbf{c}_{ia}^\dagger) = p(a) + p(i) \pmod{2}$, $p(\mathbf{n}_{ia}) = 0$. They obey the following defining relations:

$$[\mathbf{c}_{ai}, \mathbf{c}_{jb}^\dagger]_{q^{(-1)^{p(a)}\delta_{ab}\delta_{ij}}} = \delta_{ab}\delta_{ij}q^{(-1)^{p(i)}\mathbf{n}_{ia}}, \quad [\mathbf{c}_{ai}, \mathbf{c}_{jb}^\dagger]_{q^{(-1)^{p(a)}\delta_{ab}\delta_{ij}}} = \delta_{ab}\delta_{ij}q^{(-1)^{p(i)}\mathbf{n}_{ia}}, \quad (3.39)$$

$$[\mathbf{n}_{ia}, \mathbf{c}_{bj}] = -\delta_{ij}\delta_{ab}\mathbf{c}_{bj}, \quad [\mathbf{n}_{ia}, \mathbf{c}_{jb}^\dagger] = \delta_{ij}\delta_{ab}\mathbf{c}_{jb}^\dagger, \quad [\mathbf{n}_{ia}, \mathbf{n}_{jb}] = [\mathbf{c}_{ai}, \mathbf{c}_{bj}] = [\mathbf{c}_{ia}^\dagger, \mathbf{c}_{jb}^\dagger] = 0, \quad (3.40)$$

where $i, j \in I, a, b \in \bar{I}$. From (3.39), we can derive the relations: $\mathbf{c}_{ai}\mathbf{c}_{ia}^\dagger = [\mathbf{n}_{ia} + 1]_q$, $\mathbf{c}_{ia}^\dagger\mathbf{c}_{ai} = [\mathbf{n}_{ia}]_q$ for $p(i) + p(a) = 0 \pmod{2}$, and $\mathbf{c}_{ai}\mathbf{c}_{ia}^\dagger = [1 - \mathbf{n}_{ia}]_q$, $\mathbf{c}_{ia}^\dagger\mathbf{c}_{ai} = [\mathbf{n}_{ia}]_q$ for $^3 p(i) + p(a) = 1 \pmod{2}$, where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. For the diagonal part,

³We consider these generators on the Fock space fixed by the vacuum (3.73). Then for the fermionic case $p(i) + p(a) = 1 \pmod{2}$, these relation becomes $\mathbf{c}_{ai}\mathbf{c}_{ia}^\dagger = 1 - \mathbf{n}_{ia}$, $\mathbf{c}_{ia}^\dagger\mathbf{c}_{ai} = \mathbf{n}_{ia}$.

we consider the following

$$L_{ii} = q^{-(-1)^{p(i)} \sum_{b \in \bar{I}} \mathbf{n}_{ib}} \quad \text{for } i \in I, \quad (3.41)$$

$$L_{aa} = q^{(-1)^{p(a)} \sum_{j \in I} \mathbf{n}_{ja}} \quad \text{for } a \in \bar{I}, \quad (3.42)$$

$$\bar{L}_{ii} = q^{(-1)^{p(i)} \sum_{b \in \bar{I}} \mathbf{n}_{ib}} \quad \text{for } i \in I, \quad (3.43)$$

Let us look for q-oscillator realization of the non-diagonal part, which are compatible with the defining relations with the diagonal part (3.41)-(3.43). Let us introduce notations $\mathbf{n}_{[i,j],a} = \sum_{k=i}^j \mathbf{n}_{k,a}$, $\mathbf{n}_{i,[a,b]} = \sum_{c=a}^b \mathbf{n}_{i,c}$, $\mathbf{n}_{I,a} = \sum_{k \in I} \mathbf{n}_{k,a}$, $\mathbf{n}_{i,\bar{I}} = \sum_{c \in \bar{I}} \mathbf{n}_{i,c}$. We find the following solutions.

(i) The case $I = \emptyset$, $\bar{I} = \{1, 2, \dots, M + N\}$: for $a, b \in \bar{I}$,

$$L_{ab} = 0 \quad \text{for } a \neq b \quad \text{and} \quad L_{aa} = 1, \quad (3.44)$$

$$\bar{L}_{ab} = 0. \quad (3.45)$$

(ii) The case $I = \{i\}$, $\bar{I} = \{1, 2, \dots, M + N\} \setminus \{i\}$:

$$L_{\alpha\beta} = 0 \quad \text{for } \alpha < \beta \quad \text{or} \quad 1 \leq \beta < i < \alpha \leq M + N, \quad (3.46)$$

$$L_{ii} = q^{-(-1)^{p(i)} \mathbf{n}_{i,\bar{I}}}, \quad (3.47)$$

$$L_{aa} = q^{(-1)^{p(a)} \mathbf{n}_{i,a}} \quad \text{for } a \in \bar{I}, \quad (3.48)$$

$$L_{ai} = (-1)^{p(a)} \mathbf{c}_{ai} q^{(-1)^{p(i)} \mathbf{n}_{i,[i+1,a-1]}} \quad \text{for } i + 1 \leq a \leq M + N, \quad (3.49)$$

$$L_{ib} = (q - q^{-1}) \mathbf{c}_{ib}^\dagger q^{(-1)^{p(i)} \mathbf{n}_{i,[b,i-1]}} \quad \text{for } 1 \leq b \leq i - 1, \quad (3.50)$$

$$L_{ab} = (-1)^{(p(a)+p(b))(p(a)+p(i))+p(i)} (q - q^{-1}) \mathbf{c}_{ai} \mathbf{c}_{ib}^\dagger q^{(-1)^{p(i)} \mathbf{n}_{i,[b,a-1]}} \\ \text{for } 1 \leq b < a \leq i - 1 \quad \text{or} \quad i + 1 \leq b < a \leq M + N, \quad (3.51)$$

$$\bar{L}_{\alpha\beta} = 0 \quad \text{for } \alpha > \beta \quad \text{or} \quad 1 \leq \alpha \leq \beta \leq i - 1 \quad \text{or} \quad i + 1 \leq \alpha \leq \beta \leq M + N, \quad (3.52)$$

$$\bar{L}_{ii} = q^{(-1)^{p(i)} \mathbf{n}_{i,\bar{I}}}, \quad (3.53)$$

$$\bar{L}_{ai} = (-1)^{p(a)} \mathbf{c}_{ai} q^{(-1)^{p(i)} (\mathbf{n}_{i,[1,a-1]} + \mathbf{n}_{i,[i+1,M+N]})} \quad \text{for } 1 \leq a \leq i - 1, \quad (3.54)$$

$$\bar{L}_{ib} = (q - q^{-1}) \mathbf{c}_{ib}^\dagger q^{(-1)^{p(i)} (\mathbf{n}_{i,[1,i-1]} + \mathbf{n}_{i,[b,M+N]})} \quad \text{for } i + 1 \leq b \leq M + N, \quad (3.55)$$

$$\bar{L}_{ab} = (-1)^{(p(a)+p(b))(p(a)+p(i))+p(i)} (q - q^{-1}) \mathbf{c}_{ai} \mathbf{c}_{ib}^\dagger q^{(-1)^{p(i)} (\mathbf{n}_{i,[1,a-1]} + \mathbf{n}_{i,[b,M+N]})} \\ \text{for } 1 \leq a < i < b \leq M + N. \quad (3.56)$$

(iii) The case $I = \{1, 2, \dots, M + N\} \setminus \{a\}$, $\bar{I} = \{a\}$:

$$L_{\alpha\beta} = 0 \quad \text{for } \alpha < \beta, \quad (3.57)$$

$$L_{aa} = q^{(-1)^{p(a)} \mathbf{n}_{I,a}}, \quad (3.58)$$

$$L_{ii} = q^{-(-1)^{p(i)}\mathbf{n}_{i,a}} \quad \text{for } i \in I, \quad (3.59)$$

$$L_{ia} = (-1)^{p(a)}(q - q^{-1})\mathbf{c}_{ia}^\dagger q^{(-1)^{p(a)}(\mathbf{n}_{[1,a-1],a} + \mathbf{n}_{[i+1,M+N],a})} \quad \text{for } a+1 \leq i \leq M+N, \quad (3.60)$$

$$L_{aj} = q^{-(-1)^{p(a)}\mathbf{c}_{aj}q^{(-1)^{p(a)}(\mathbf{n}_{[1,j],a} + \mathbf{n}_{[a+1,M+N],a})}} \quad \text{for } 1 \leq j \leq a-1, \quad (3.61)$$

$$L_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)+1}(q - q^{-1})\mathbf{c}_{ia}^\dagger \mathbf{c}_{aj} q^{(-1)^{p(a)}\mathbf{n}_{[j+1,i],a}} \\ \text{for } 1 \leq j < i \leq a-1 \quad \text{or} \quad a+1 \leq j < i \leq M+N, \quad (3.62)$$

$$L_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)} q^{-(-1)^{p(a)}}(q - q^{-1})\mathbf{c}_{ia}^\dagger \mathbf{c}_{aj} q^{(-1)^{p(a)}(\mathbf{n}_{[1,j],a} + \mathbf{n}_{[i+1,M+N],a})} \\ \text{for } 1 \leq j < a < i \leq M+N, \quad (3.63)$$

$$\bar{L}_{\alpha\beta} = 0 \quad \text{for } \alpha > \beta \quad \text{or} \quad \alpha = \beta = a, \quad (3.64)$$

$$\bar{L}_{ii} = q^{(-1)^{p(i)}\mathbf{n}_{i,a}} \quad \text{for } i \in I, \quad (3.65)$$

$$\bar{L}_{ia} = (-1)^{p(a)}(q - q^{-1})\mathbf{c}_{ia}^\dagger q^{(-1)^{p(a)}\mathbf{n}_{[i+1,a-1],a}} \quad \text{for } 1 \leq i \leq a-1, \quad (3.66)$$

$$\bar{L}_{aj} = q^{-(-1)^{p(a)}\mathbf{c}_{aj}q^{(-1)^{p(a)}\mathbf{n}_{[a+1,j],a}}} \quad \text{for } a+1 \leq j \leq M+N, \quad (3.67)$$

$$\bar{L}_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)+1}(q - q^{-1})\mathbf{c}_{ia}^\dagger \mathbf{c}_{aj} q^{(-1)^{p(a)}(\mathbf{n}_{[1,i],a} + \mathbf{n}_{[j+1,M+N],a})} \\ \text{for } 1 \leq i < a < j \leq M+N, \quad (3.68)$$

$$\bar{L}_{ij} = (-1)^{(p(i)+p(j))p(a)+p(i)p(j)} q^{-(-1)^{p(a)}}(q - q^{-1})\mathbf{c}_{ia}^\dagger \mathbf{c}_{aj} q^{(-1)^{p(a)}\mathbf{n}_{[i+1,j],a}} \\ \text{for } 1 \leq i < j \leq a-1 \quad \text{or} \quad a+1 \leq i < j \leq M+N. \quad (3.69)$$

(iv) The case $I = \{1, 2, \dots, M+N\}$, $\bar{I} = \emptyset$: for $i, j \in I$,

$$L_{ij} = \bar{L}_{ij} = 0 \quad \text{for } i \neq j \quad \text{and} \quad L_{ii} = \bar{L}_{ii} = 1. \quad (3.70)$$

The q-oscillator solutions of the graded Yang-Baxter equation are given by substituting the above q-oscillator realizations of the L-operators into the map (3.27). We denote the corresponding solutions as

$$\mathbf{L}_I(x) = \mathbf{L} - \bar{\mathbf{L}}x^{-1}. \quad (3.71)$$

We remark that the following renormalized L-operators

$$\mathcal{L}_I(v) := 1 \otimes \left(\sum_{i \in I} (q - q^{-1})^{-1} E_{ii} + \sum_{b \in \bar{I}} E_{bb} \right) q^v \mathbf{L}_I(q^{2v}), \quad v \in \mathbb{C} \quad (3.72)$$

reduce to L-operators similar to the ones in [17] in the rational limit $q \rightarrow 1$.

Now (3.71) defines an evaluation map from the q-superYangian to the contracted algebra. Let us calculate the actions of generators on the vacuum defined by

$$\mathbf{n}_{ia}|0\rangle = \mathbf{c}_{ia}|0\rangle = 0 \quad \text{for all } i \in I, a \in \bar{I}. \quad (3.73)$$

They lead

$$\begin{aligned} L_{ii}(x)|0\rangle &= (1 - x^{-1})|0\rangle \quad \text{for } i \in I, \\ L_{aa}(x)|0\rangle &= |0\rangle \quad \text{for } a \in \bar{I}. \end{aligned} \quad (3.74)$$

In particular for $I = \{1, 2, \dots, n\} \subset \{1, 2, \dots, M + N\}$, we find

$$L_{ij}(x)|0\rangle = 0 \quad \text{for } i > j. \quad (3.75)$$

Thus the corresponding representation is the highest weight representations of the q -superYangian with the highest weight vector $|0\rangle$ and the highest weight given by (3.74). In addition, the ratio of the eigenvalues $\nu_i(x)$ of $L_{ii}(x)$ on $|0\rangle$ is $\nu_i(x)/\nu_{i+1}(x) = 1 - x^{-1}\delta_{n,i}$ for $1 \leq i \leq M + N - 1$. This is a kind of Drinfeld rational fraction⁴ introduced in [15]. The finite dimensional representations of the quantum affine algebras are characterized by the Drinfeld polynomials. In contrast, q -oscillator representations given as limits of the Kirillov-Reshetikhin modules⁵ of the Borel subalgebra of the quantum affine algebras are characterized by the Drinfeld rational fractions. For the other sets I , the highest weight condition (3.75) is changed as they should be interpreted as representations permuted by the Weyl group. Let us consider a L -operator $\tilde{\mathbf{L}}(x) = \mathbf{L}(xq^{-2m})(1 \otimes q^{-m} \sum_{i \in I} E_{ii})$ for the q -superYangian shifted by the automorphisms (3.11) and (3.12). For an evaluation representation based on the map (3.27) and the highest weight representation of $U_q(gl(M|N))$ with the highest weight $\lambda_i = (-1)^{p(i)}m$ for $i \in I$ and $\lambda_a = 0$ for $a \in \bar{I}$ (cf. (3.31)), the eigenvalues of the diagonal part of $\tilde{\mathbf{L}}(x)$ on the highest weight vector coincides with the ones in (3.74) in the limit $m \rightarrow \infty$ ($|q| < 1$).

The evaluation map (2.20) has another presentation of the form:

$$\begin{aligned} e_0 &\mapsto -(-1)^{p(1)}x(q - q^{-1})^{-1}\bar{L}_{1,M+N}L_{M+N,M+N}^{-1}, \\ f_0 &\mapsto x^{-1}(q - q^{-1})^{-1}L_{M+N,M+N}L_{M+N,1}, \\ h_0 &\mapsto \frac{\log(L_{M+N,M+N}L_{1,1}^{-1})}{\log q}, \\ e_i &\mapsto (-1)^{p(i+1)}(q - q^{-1})^{-1}L_{i+1,i}L_{ii}^{-1}, \\ f_i &\mapsto -(q - q^{-1})^{-1}L_{ii}\bar{L}_{i,i+1}, \\ h_i &\mapsto \frac{\log(L_{ii}L_{i+1,i+1}^{-1})}{\log q} \quad \text{for } 1 \leq i \leq M + N - 1. \end{aligned} \quad (3.76)$$

Let us substitute L_{ij} given by (3.41)-(3.70) (for a fixed I) into the right hand side of (3.76). This gives evaluation map from \mathcal{B}_+ or \mathcal{B}_- to the q -oscillator superalgebra. We

⁴Here the spectral parameter x came from \mathcal{B}_- . To interpret it as the one from \mathcal{B}_+ , we have to replace x with x^{-1} .

⁵The q -characters or the T-functions for the Kirillov-Reshetikhin modules solve the T-system for $MN = 0$ [31] and for $MN \neq 0$ [32].

denote this map as $\rho_I(x)$. Similar maps from \mathcal{B}_+ to the q-oscillator (super)algebra were considered for $U_q(\hat{sl}(2))$ [5], $U_q(\hat{sl}(3))$ [6], $U_q(\hat{sl}(M))$ [9] and $U_q(\hat{sl}(2|1))$ [1]. Here we used L_{ii}^{-1} instead of \overline{L}_{ii} since L_{ii}^{-1} do not coincide with \overline{L}_{ii} for the contracted algebras on $U_q(gl(M|N))$. We remark that $\rho_I(x)$ is not an evaluation map from $U_q(\hat{sl}(M|N))$ to the q-oscillator superalgebra but rather a map from a contracted algebra of $U_q(\hat{sl}(M|N))$. In fact, the following contracted commutation relations hold true under the map.

$$[e_i, f_j] = \begin{cases} \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} & \text{for } i, i+1 \in I, \\ \frac{\delta_{ij} q^{h_i}}{q - q^{-1}} & \text{for } i \in \overline{I}, \\ \frac{-\delta_{ij} q^{-h_i}}{q - q^{-1}} & \text{for } i+1 \in \overline{I}, \\ 0 & \text{for } i, i+1 \in \overline{I}, \end{cases} \quad (3.77)$$

where i, j should be interpreted under $\text{mod } M + N$. The other commutation relations are basically the same as the ones in section 2. Some of the relations becomes $0 = 0$ when the generator f_i vanishes.

Our L-operators (3.71) satisfy the defining relations of the universal R-matrix. In particular, the following relations are valid

$$\begin{aligned} (1 \otimes \pi(y)(h_i) + \rho_I(x)(h_i) \otimes 1) \mathbf{L}_I(y/x) = \\ = \mathbf{L}_I(y/x) (\rho_I(x)(h_i) \otimes 1 + 1 \otimes \pi(y)(h_i)), \end{aligned} \quad (3.78)$$

$$\begin{aligned} (1 \otimes \pi(y)(e_i) + \rho_I(x)(e_i) \otimes \pi(y)(q^{-h_i})) \mathbf{L}_I(y/x) = \\ = \mathbf{L}_I(y/x) (\rho_I(x)(e_i) \otimes 1 + \rho_I(x)(q^{-h_i}) \otimes \pi(y)(e_i)), \end{aligned} \quad (3.79)$$

where $0 \leq i \leq M + N - 1$. This is because our L-operators are image of the universal R-matrix (up to an overall factor $N_I(x, y)$): $\mathbf{L}_I(y/x) = N_I(x, y)(\rho_I(x) \otimes \pi(y))(\mathcal{R})$ (see also discussions on the universal R-matrix in [33]). Note that

$$\begin{aligned} (\rho_I(x)(q^{h_i}) \otimes \pi(y)(f_i) + \rho_I(x)(f_i) \otimes 1) \mathbf{L}_I(y/x) = \\ = \mathbf{L}_I(y/x) (\rho_I(x)(f_i) \otimes \pi(y)(q^{h_i}) + 1 \otimes \pi(y)(f_i)) \end{aligned} \quad (3.80)$$

hold true only for the case $i, i+1 \in I$ ($0 \equiv M + N$) as we are considering a contracted algebra.

In this paper, we consider contractions defined by (3.35)-(3.36). Instead of (3.36), one can consider the following:

$$L_{ii} = 0 \quad \text{for } i \in \overline{I}. \quad (3.81)$$

The L-operators based on this contraction have one to one correspondence to the ones proposed in this paper. They seem to be the image of the Cartan anti-involution

for our L-operators. One may also consider more general contractions than (3.36) and (3.81):

$$L_{ii} = 0 \quad \text{for } i \in \overline{I}_1, \quad \overline{L}_{ii} = 0 \quad \text{for } i \in \overline{I}_2, \quad \overline{I}_1, \overline{I}_2 \subset \overline{I}. \quad (3.82)$$

This defines more degenerated algebras and gives degenerated solutions of the graded Yang-Baxter equation.

4 T- and Q-operators

In this section, we define Q-operators based on the q-oscillator representations introduced in the previous section and sketch how to write the T-operators in terms of them. This gives a cue for operator realization of the formulas in our previous papers [2, 3].

We introduce the universal boundary operator

$$\mathcal{D} = q^{\sum_{k=1}^{M+N-1} \sum_{i=1}^k (-1)^{p(i)} \varphi_i h_k}, \quad (4.1)$$

where $\varphi_i \in \mathbb{C}$. We also define a parameter φ_{M+N} by the relation $\sum_{i=1}^{M+N} (-1)^{p(i)} \varphi_i = 0$. This boundary operator is a Cartan element of $U_q(\hat{sl}(M|N))$. Due to the first relation in (2.14), its co-product commutes with the universal R-matrix

$$\mathcal{R}(\mathcal{D} \otimes \mathcal{D}) = (\mathcal{D} \otimes \mathcal{D}) \mathcal{R}. \quad (4.2)$$

The images of the evaluation map (2.20) and $\rho_I(x)$ are given as

$$\mathbf{D} := \mathbf{ev}_x(\mathcal{D}) = q^{\sum_{i=1}^{M+N} \varphi_i e_{ii}}, \quad (4.3)$$

$$\mathbf{D}_I := \rho_I(x)(\mathcal{D}) = q^{\sum_{i \in I, a \in \overline{I}} (\varphi_i - \varphi_a) \mathbf{n}_{ia}}. \quad (4.4)$$

We define the universal T-operator by

$$\mathbb{T}_\lambda(x) = (\text{Str}_{\pi_\lambda(x)} \otimes 1) [\mathcal{R}(\mathcal{D} \otimes 1)]. \quad (4.5)$$

Note that $\mathbb{T}_\lambda(x)$ is an element of \mathcal{B}_- and this definition of the T-operator does not depend on the particular representation of the quantum space. It is convenient to introduce operators

$$z_k = q^{\varphi_k + (-1)^{p(k)} \sum_{j=1}^{M+N-1} (d_{kj} - d_{k-1,j}) h_j}, \quad (4.6)$$

where $1 \leq k \leq M+N$ and $d_{k0} = d_{M+N,j} = 0$. This satisfies $\prod_{k=1}^{M+N} z_k^{(-1)^{p(k)}} = 1$. Then the T-operator (4.5) can be rewritten as

$$\mathbb{T}_\lambda(x) = (\text{Str}_{\pi_\lambda(x)} \otimes 1) [\overline{\mathcal{R}} \overline{\mathcal{D}}], \quad (4.7)$$

where

$$\overline{\mathcal{D}} := q^{\mathcal{K}}(\mathcal{D} \otimes 1) = \prod_{k=1}^{M+N-1} \left(\prod_{i=1}^k (1 \otimes z_i^{(-1)^{p(i)}}) \right)^{h_k \otimes 1}. \quad (4.8)$$

Here we have renormalized the boundary operator (4.1) by the prefactor of the universal R-matrix (2.16) as in [1]. If there is no reduced universal R-matrix in (4.7), the following quantity

$$\mathbb{Z}(\lambda) = (\text{Str}_{\pi_\lambda(x)} \otimes 1) [\overline{\mathcal{D}}], \quad (4.9)$$

gives the supercharacter. For finite dimensional modules, it is a supersymmetric Schur function on the variables (4.6). In particular for the Verma module, it leads

$$\mathbb{Z}^+(\lambda) := (\text{Str}_{\pi_\lambda^+(x)} \otimes 1) [\overline{\mathcal{D}}] = \frac{\prod_{j=1}^M z_j^{\lambda_j + M - N - j} \prod_{k=M+1}^{M+N} (-z_k)^{\lambda_k + N - M - k}}{\mathbf{D}}, \quad (4.10)$$

$$\mathbf{D} := \frac{\prod_{1 \leq b < b' \leq M} (z_b - z_{b'}) \prod_{M+1 \leq f < f' \leq M+N} (z_{f'} - z_f)}{\prod_{b=1}^M \prod_{f=M+1}^{M+N} (z_b - z_f)}. \quad (4.11)$$

In the above formulas, the reduced universal R-matrix plays a role to put the spectral parameter into the supercharacters, or to change the supercharacters to the q-supercharacters. This induces sort of shifts on the parameters (4.6) in the supercharacters. Let \mathcal{F}_I be the Fock space defined by the action of the generators $\{\mathbf{c}_{ai}, \mathbf{c}_{ia}^\dagger, \mathbf{n}_{ia}\}$ ($i \in I, a \in \overline{I}$) of the q-oscillator superalgebras on the vacuum (3.73). We define the universal Q-operator by

$$\mathbb{Q}_I(x) = \mathbb{Z}_I^{-1}(\text{Str}_{\mathcal{F}_I} \otimes 1)(\rho_I(x) \otimes 1) [\overline{\mathcal{R}} \overline{\mathcal{D}}], \quad (4.12)$$

where the normalization function reads

$$\mathbb{Z}_I = (\text{Str}_{\mathcal{F}_I} \otimes 1)(\rho_I(x) \otimes 1) [\overline{\mathcal{D}}]. \quad (4.13)$$

Note that these are elements of \mathcal{B}_- . We remark that (4.12) is basically fixed by the map $\rho_I(x)$ and the defining relations of the q-oscillator superalgebra (3.39) and does not depend on the definition of the vacuum (see section 5.2.3 in [1] for more details). Due to the graded Yang-Baxter equation (2.15) and the commutation relation (4.2), the universal T- and Q-operators are commutative.

$$\begin{aligned} \mathbb{T}_\lambda(x) \mathbb{T}_\mu(y) &= \mathbb{T}_\mu(y) \mathbb{T}_\lambda(x), & \mathbb{T}_\lambda(x) \mathbb{Q}_I(y) &= \mathbb{Q}_I(y) \mathbb{T}_\lambda(x), \\ \mathbb{Q}_I(x) \mathbb{Q}_J(y) &= \mathbb{Q}_J(y) \mathbb{Q}_I(x), \end{aligned} \quad (4.14)$$

where $x, y \in \mathbb{C}$, $I, J \subset \{1, 2, \dots, M+N\}$ and λ, μ are any highest weights.

Let us calculate the supertrace (4.13) over the Fock space \mathcal{F}_I . Explicitly, it leads

$$\mathbb{Z}_I = \prod_{i \in I} \prod_{a \in \bar{I}} \left(1 - \frac{z_a}{z_i}\right)^{-(-1)^{p(i)+p(a)}}. \quad (4.15)$$

As expected, this coincides with a limit of a normalized character of the Kirillov-Reshetikhin module at least for the case ⁶ $N = 0$ (cf. [15]):

$$\mathbb{Z}_I = \lim_{\eta \rightarrow \infty} \frac{S_\lambda(z_1, z_2, \dots, z_M)}{\prod_{k=1}^M z_k^{\lambda_k}}, \quad |z_i| > |z_a| \quad \text{for all } i \in I, a \in \bar{I},$$

$$\eta := \lambda_k \quad \text{for } k \in I, \quad \lambda_k = 0 \quad \text{for } k \in \bar{I}, \quad (4.16)$$

where $S_\lambda(z_1, z_2, \dots, z_M) = \det_{1 \leq i, j \leq M} (z_i^{M+\lambda_j-j}) / \det_{1 \leq i, j \leq M} (z_i^{M-j})$ is the Schur function. Here we meant the equality in (4.16) by the substitution of elements of \mathcal{B}_- (4.6) for the complex numbers $\{z_k\}$ on the right hand side after the limit. We expect [2, 3] that the T-operator is given by the Baxterization of the supercharacter ⁷

$$\mathbb{T}_\lambda(x) = \frac{1}{D} \prod_{k=1}^{M+N} Q_{\{k\}}(xq^{-d_k}) \cdot [D \mathbb{Z}(\lambda)] \quad (4.17)$$

where d_k are differential operators which evaluate the degrees of the monomials on $\{z_j\}$ in the right of the dot \cdot . They effectively act as $d_k = 2(-1)^{p(k)} z_k \frac{\partial}{\partial z_k}$ in $[\dots]$. We assume d_k act on the functions in the left of the dot \cdot as just an identity, although $\{Q_{\{k\}}\}$ are also functions of $\{z_k\}$. In particular for the Verma module ⁸, we have [3]

$$\mathbb{T}_\lambda^+(x) = \mathbb{Z}^+(\lambda) \prod_{j=1}^{M+N} Q_{\{j\}}(xq^{-2((-1)^{p(j)}\lambda_j - \sum_{k=1}^{j-1} (-1)^{p(k)})}). \quad (4.18)$$

We remark that the most of the T-operators can be written as summations of the above formula (4.18) based on the Bernstein-Gelfand-Gelfand resolution and rewritten as Wronskian-like determinants (see [5] for $U_q(\hat{sl}(2))$, [6] for $U_q(\hat{sl}(3))$, [9] for finite dimensional representations of $U_q(\hat{sl}(M))$ (see also a Wronskian like determinant in [34]), [1] for $U_q(\hat{sl}(2|1))$; [2, 3] for the Wronskian-like determinants for

⁶We have also checked that a normalized Sergeev-Pragacz formula produces (4.15) in the large Young diagram limit under a similar condition for the case $MN \neq 0$.

⁷The shift of the spectral parameter of the Q-operators in [2, 3] can be recovered by putting $q \rightarrow q^{-1}$ after the replacement $Q_I(x) \mapsto Q_I(xq^{\sum_{k \in I} (-1)^{p(k)}})$.

⁸This formula (4.18) was presented first as a poster at a conference ‘Integrability in Gauge and String Theory 2010’, Nordita, Sweden, 28 June 2010 - 2 July. To fit the formula in [3], one has to make an overall shift of the spectral parameter $x \rightarrow xq^{2(M-N)}$ after the manipulation in the footnote 7.

any $U_q(\hat{gl}(M|N))$. We expect our universal Q-operators obey functional relations of the form: for $p(i) = p(j)$:

$$\begin{aligned} (z_i - z_j) \mathbb{Q}_I(xq^{1-2p(i)}) \mathbb{Q}_{I \cup \{i,j\}}(xq^{-1+2p(i)}) = \\ = z_i \mathbb{Q}_{I \cup \{i\}}(xq^{-1+2p(i)}) \mathbb{Q}_{I \cup \{j\}}(xq^{1-2p(i)}) - z_j \mathbb{Q}_{I \cup \{i\}}(xq^{1-2p(i)}) \mathbb{Q}_{I \cup \{j\}}(xq^{-1+2p(i)}), \end{aligned} \quad (4.19)$$

and for $p(i) \neq p(j)$:

$$\begin{aligned} (z_i - z_j) \mathbb{Q}_{I \cup \{i\}}(xq^{-1+2p(i)}) \mathbb{Q}_{I \cup \{j\}}(xq^{1-2p(i)}) = \\ = z_i \mathbb{Q}_I(xq^{1-2p(i)}) \mathbb{Q}_{I \cup \{i,j\}}(xq^{-1+2p(i)}) - z_j \mathbb{Q}_I(xq^{-1+2p(i)}) \mathbb{Q}_{I \cup \{i,j\}}(xq^{1-2p(i)}). \end{aligned} \quad (4.20)$$

At the moment, these functional relations are fully proven for $U_q(\hat{sl}(2))$ [5], for $U_q(\hat{sl}(3))$ [6] and for $U_q(\hat{sl}(2|1))$ [1]. Their proof is based on decompositions of q-oscillator representations of \mathcal{B}_+ and does not rely on the representation of \mathcal{B}_- on the quantum space. See also [20, 17] for discussions on rational models. On the level of the eigenvalues of Q-operators for rational models, (4.20) were discussed in details in relation to the Bäcklund transformations [35]. Here we used expressions based on the 2^{M+N} index sets on the Hasse diagram presented in [2].

Now that we have the universal T- and Q-operators (4.5), (4.12), our next task is to evaluate these for particular representations of \mathcal{B}_- on the quantum space of the model. For example, the T-operator for the lattice model whose quantum space is the fundamental representation on each site is given as

$$\mathbf{T}_\lambda(x) = N_\lambda^{(L)}(x) (\pi(\xi_1) \otimes \pi(\xi_2) \cdots \otimes \pi(\xi_L)) [\Delta^{(L-1)} \mathbb{T}_\lambda(x)] \quad (4.21)$$

$$= \text{Str}_{\pi_\lambda} [\mathbf{L}^{0L}(\xi_L/x) \cdots \mathbf{L}^{02}(\xi_2/x) \mathbf{L}^{01}(\xi_1/x) (\mathbf{D} \otimes 1^{\otimes L})], \quad (4.22)$$

where L is the number of the lattice site, the complex parameters $\{\xi_j\}_{j=1}^L$ are inhomogeneities on the spectral parameter and $N_\lambda^{(L)}(x)$ is a function for the normalization. In (4.22), the evaluation map (3.27) is used and the supertrace is taken over the auxiliary space denoted as ‘0’. The Q-operators for the same system are given by

$$\mathbf{Q}_I(x) = N_I^{(L)}(x) (\pi(\xi_1) \otimes \pi(\xi_2) \cdots \otimes \pi(\xi_L)) [\Delta^{(L-1)} \mathbb{Q}_I(x)] \quad (4.23)$$

$$= \mathbf{Z}_I^{-1} \text{Str}_{\mathcal{F}_I} [\mathbf{L}_I^{0L}(\xi_L/x) \cdots \mathbf{L}_I^{02}(\xi_2/x) \mathbf{L}_I^{01}(\xi_1/x) (\mathbf{D}_I \otimes 1^{\otimes L})], \quad (4.24)$$

where $\mathbf{Z}_I := (\pi(\xi_1) \otimes \pi(\xi_2) \cdots \otimes \pi(\xi_L)) [\Delta^{(L-1)} \mathbb{Z}_I]$ and the normalization function is $N_I^{(L)}(x) := \prod_{k=1}^L N_I(x, \xi_k)$. It is instructive to calculate the lattice T-operator (4.22) for the Verma module ⁹ and the lattice Q-operator (4.24) even for one site

⁹We remark that a formula similar to the first equality in (4.25) (for the characters of finite dimensional representations of $U_q(gl(M))$) was previously derived by Anton Zabrodin in 2007 based on the trigonometric version of the co-derivative for $L = 1$ case.

$L = 1$ case. Let us introduce a notation $\mathbf{Z}^+(\lambda) := \pi(\xi_1)(\mathbb{Z}^+(\lambda))$. Then we obtain

$$\begin{aligned}
[\mathbf{T}_\lambda^+(x)]_{ii} &= \mathbf{Z}^+(\lambda) - \frac{x}{\xi_1} q^{-d_i} \cdot \mathbf{Z}^+(\lambda) = \\
&= \mathbf{Z}^+(\lambda) \left(1 - \frac{x q^{-2((-1)^{p(i)}\lambda_i - \sum_{k=1}^{i-1} (-1)^{p(k)})}}{\xi_1} \prod_{\substack{b=1 \\ b \neq i}}^{M+N} \left(\frac{1 - \frac{z_b}{z_i}}{1 - \frac{z_b q^{2(-1)^{p(i)}}}{z_i}} \right)^{(-1)^{p(i)+p(b)}} \right) \\
&\quad \text{for } 1 \leq i \leq M+N, \\
[\mathbf{T}_\lambda^+(x)]_{\alpha\beta} &= 0 \quad \text{for } \alpha \neq \beta,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
[\mathbf{Q}_I(x)]_{ii} &= 1 - \frac{x}{\xi_1} \frac{q^{-d_i} \cdot \mathbf{Z}_I}{\mathbf{Z}_I} = 1 - \frac{x}{\xi_1} \prod_{b \in \bar{I}} \left(\frac{1 - \frac{z_b}{z_i}}{1 - \frac{z_b q^{2(-1)^{p(i)}}}{z_i}} \right)^{(-1)^{p(i)+p(b)}} \quad \text{for } i \in I, \\
[\mathbf{Q}_I(x)]_{aa} &= 1 \quad \text{for } a \in \bar{I}, \\
[\mathbf{Q}_I(x)]_{\alpha\beta} &= 0 \quad \text{for } \alpha \neq \beta,
\end{aligned} \tag{4.26}$$

where $[\mathbf{M}]_{\alpha\beta}$ denotes the (α, β) matrix element of a $((M+N) \times (M+N))$ matrix \mathbf{M} . In (4.25) and (4.26), the twist parameters should be interpreted as (i, i) -matrix element of them:

$$z_k = [z_k]_{ii} = q^{\varphi_k + (-1)^{p(k)+p(i)}(d_{ki} - d_{k-1,i} - d_{k,i-1} + d_{k-1,i-1})}. \tag{4.27}$$

The above example gives a non-trivial support to the QQ-relations (4.19)-(4.20) and the factorization formulas (4.18) for the Verma module as the shape of these equations will be essentially independent of the quantum space of the model. This also agrees with examples in eqs. (3.38)-(3.43) in [1] up to a transformation $q \rightarrow q^{-1}$ and a rescaling of the spectral parameter.

The other interesting examples of the Q-operators are the ones for the conformal field theory (CFT). The monodromy matrix of the CFT can be expressed as an ordered exponential of the form $\bar{\mathcal{L}} = \mathcal{P} \exp \left(\sum_{i=0}^{M+N-1} \int_0^{2\pi} du e_i \otimes V_i(u) \right)$, where $V_i(u)$ are q-vertex operators obeying $V_i(u)V_j(v) = (-1)^{p(i)p(j)} q^{c_{ij}} V_j(v)V_i(u)$ for $u > v$ (c_{ij} : a symmetrized Cartan matrix of $sl(M|N)$) and e_i are the generators of \mathcal{B}_+ . Thus, if we substitute our q-oscillator realizations of \mathcal{B}_+ through (3.76) into the formula and taking the supertrace over the Fock space for \mathcal{B}_+ , we will obtain Q-operators for the CFT. Examples of such Q-operators can be seen for $(M, N) = (2, 0)$ in [5], $(M, N) = (3, 0)$ in [6], $N = 0$ in [9], $(M, N) = (2, 1)$ in [1] and for $U_q(C(2)^{(2)})$ in [7]. See also a related recent paper [33].

Finally, we can define the universal master T-operator [21] by

$$\tau(x, t) = \sum_{\lambda} S_{\lambda}(t) \mathbb{T}_{\lambda}(x), \quad (4.28)$$

where $t = (t_1, t_2, \dots)$ are time variables in the KP hierarchy and $S_{\lambda}(t)$ is the Schur function labeled by the Young diagram λ . This is a τ -function of the modified KP hierarchy and allows embedding of the quantum integrable system into the soliton theory. Basically, all the functional relations among T- and Q-operators in the Hirota form can be derived from this (see [21, 20] for more details).

5 Concluding remarks

In this paper, we have developed our preliminary discussions on L-operators for the Baxter Q-operators for $U_q(\widehat{sl}(2|1))$ [14, 1] and $U_q(\widehat{gl}(3))$ [16], and generalized them to the higher rank case $U_q(\widehat{gl}(M|N))$. The model independent universal Q-operators are defined as supertrace of the universal R-matrix. This is a step toward our trial [1, 2, 3] (also [20, 21]) to construct systematically Q-operators and Wronskian-like expressions of T-operators in terms of them. The L-operators given in this paper can be building blocks of them. Our next task [36] directly related to this paper will be mainly two fold: to generalize our q-oscillator realization of the L-operators for the Q-operators to all the intermediate ones labeled by any 2^{M+N} index set I , and to generalize these for more general representations on the quantum space. All these will be basically accomplished by evaluating the universal R-matrix in the light of asymptotic representations of the quantum affine algebra [15]. A fusion method [17, 19] on L-operators for Q-operators developed for rational models may also be helpful for this.

A generalization to the elliptic case is perhaps interesting. Although whether the contraction of the Sklyanin algebra works is not clear at the moment, elliptic L-operators may be given by twists of our trigonometric L-operators since the elliptic algebras (for both vertex type models and face type models) can be obtained by twists on the quantum affine algebras [37].

The other obvious direction of further development will be a generalization to the other quantum affine superalgebras. For this, it will be helpful to characterize our L-operators as sort of Lax operators for the generalized Toda system [38] in terms of the asymptotic algebra [15] and investigate the system in the light of the soliton theory [20, 21].

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¹⁰ We were pointed out by Carlo Meneghelli that our L-operators pose a problem in the rational limit. We found that this is due to the following misprint. In page 465, eq. (2.86), $C_f = q^{-1}q^{-2\mathcal{H}_f}(1 - f^+f^-)$ is a misprint of $C_f = qq^{-2\mathcal{H}_f}(f^+f^- + q^{-2}f^-f^+) = q^{-1}q^{-2\mathcal{H}_f}(1 + (q^2 - 1)f^+f^-)$. We thank him for his kind comment.

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